International Journal of Theoretical Physics, Vol. 32, No. 5, 1993

Relativistic Quantum Physics Equation for Number of Electrons

A. S. Rabinowitch[1]

Received June 18, 1992

A generalization of the Dirac equation for number of electrons is proposed which in the nonrelativistic case takes the form of the corresponding Schrödinger equation. The equivalence of various matrix representations and the relativistic covariance of the proposed equation are proved.

Let us assume N electrons are situated closely together in space. To describe them we introduce a wave function ψ with 4^N components ψ_k. We shall seek a generalization of the Dirac equation in the following form:

$$\left[\Gamma^k\left(i\hbar\frac{\partial}{\partial x^k}-\frac{e}{c}A_k\right)-\sqrt{N}\,mc\right]\psi=0 \tag{1}$$

where e, m are the charge and mass of the electron, A_k are the electromagnetic potentials of both the external fields and the interaction of the electrons, Γ^k are square matrixes of order 4^N, $0\leq k\leq 4N-1$, and x^k $[4(l-1)\leq k\leq 4l-1]$ are the coordinates of the lth particle.

We impose the following correlations on the matrixes Γ^k:

$$\Gamma^n\Gamma^m+\Gamma^m\Gamma^n=2g^{nm}E \tag{2}$$

where $g^{nm}=0$, $n\neq m$; $g^{nn}=1$, $n=4l$; $g^{nn}=-1$, $n\neq 4l$; and E is the unit matrix.

Then, after multiplying (1) by $\{\Gamma^k[i\hbar\,\partial_k-(e/c)A_k]+\sqrt{N}\,mc\}$ we derive the generalization of the Klein–Gordon equation:

$$\left[g^{kn}\left(i\hbar\,\partial_k-\frac{e}{c}A_k\right)\left(i\hbar\,\partial_n-\frac{e}{c}A_n\right)-\frac{i\hbar e}{c}\Gamma^k\Gamma^n F_{kn}-Nm^2c^2\right]\psi=0$$
$$F_{kn}=\partial_k A_n-\partial_n A_k \tag{3}$$

[1]Bryanskaya Ulitza, 12-63 Moscow 121059, Russia.

0020-7748/93/0500-0791$07.00/0

Let us consider the nonrelativistic stationary case:

$$\psi(x^k)=\exp\left\{-\frac{i}{\hbar c}\left[(mc^2+\mathscr{E}_1)x^0+(mc^2+\mathscr{E}_2)x^4+\cdots+(mc^2+\mathscr{E}_N)x^{4(N-1)}\right]\right\}\varphi(x^s) \tag{4}$$

$$s\neq 4l, \qquad |\mathscr{E}_k|\ll mc^2$$

and assume that $A_k=\phi_k$ with $k=4l$ and $A_k=0$ with $k\neq 4l$.

Then for this nonrelativistic case from (3) and (4) we obtain the Schrödinger equation for N electrons:

$$\left(\mathscr{E}-e\phi-\frac{\hbar^2}{2m}\partial^s\partial_s\right)\psi=0$$
$$s\neq 4l, \qquad \mathscr{E}=\sum_{k=1}^{N}\mathscr{E}_k, \qquad \phi=\sum_{k=1}^{N}\phi_k \tag{5}$$

The problem of solving correlations (2) is related to the spinor analysis which was investigated by Rashevsky (1955). But this method of introducing spinors based on the Clifford algebra is very complicated and besides it is not connected with any definite quantum physics equation for the N electrons. That is why we have chosen a simpler method which will permit us to generalize the well-known results of the spinor theory based on the Dirac equation (Bjorken and Drell, 1964).

Let us introduce the matrixes G^n:

$$G^n=\Gamma^n, \quad n=4l, \quad G^n=i\Gamma^n, \quad n\neq 4l$$
$$(G^n)^2=E, \quad G^nG^k=-G^kG^n, \quad n\neq k \tag{6}$$

We shall show that there are $2M$ matrixes G^n of order 2^M fulfilling (6) and that any realization of (6) is related to these matrixes by means of a similarity transformation.

Let us introduce the aggregate $\bar{G}_M$ of the following $2M$ matrixes of order 2^M by means of the recurrence method:

$$\bar{G}_k=\left(\begin{pmatrix}E_{k-1} & O_{k-1}\\ O_{k-1} & -E_{k-1}\end{pmatrix},\begin{pmatrix}O_{k-1} & E_{k-1}\\ E_{k-1} & O_{k-1}\end{pmatrix},\begin{pmatrix}O_{k-1} & i\bar{G}_{k-1}\\ -i\bar{G}_{k-1} & O_{k-1}\end{pmatrix}\right)$$
$$2\leq k\leq M; \qquad \bar{G}_1=\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) \tag{7}$$

E_{k-1} and O_{k-1} are the unit and zero matrixes of order 2^{k-1}. The last matrix in (7) means the aggregate of all of the matrixes

$$\begin{pmatrix} O_{k-1} & iF_{k-1} \\ -iF_{k-1} & O_{k-1} \end{pmatrix}$$

where $F_{k-1} \in \bar{G}_{k-1}$.

We shall prove the following theorem.

Theorem 1. The aggregate of the matrixes $\bar{G}_M$ of (7) fulfills (6) and any aggregate $\bar{H}_M$ of $2M$ matrixes fulfilling (6) can be defined by the similarity correlation $\bar{H}_M \sim \bar{G}_M$, that is, $\bar{H}_M = T^{-1}\bar{G}_M T$, where T is a nonsingular matrix.

Proof. Let us assume that G^n are the matrixes of the aggregate $\bar{H}_M$. Then, from conditions (6),

$$(G^0)^2 = E_M, \qquad G^0 = (G^1)^{-1}(-G^0)G^1 \ (G^0 \sim -G^0)$$

and it follows that the Jordan form of G^0 is the matrix

$$L^0 = \begin{pmatrix} E_{M-1} & O_{M-1} \\ O_{M-1} & -E_{M-1} \end{pmatrix}$$

Therefore we can find a similarity transformation of G^n in which G^0 is transformed into L^0. After applying this transformation which preserves the correlation (6) because it is a similarity transformation, we can consider $G^0 = L^0$ retaining the former notations for the transformed matrixes G^n. Then from (6) we derive

$$G^1 = \begin{pmatrix} O_{M-1} & S_{M-1} \\ S_{M-1}^{-1} & O_{M-1} \end{pmatrix}$$

where S_{M-1} is a nonsingular matrix of order 2^{M-1}.

Let us now apply to G^n the similarity transformation with the transformation matrix

$$T_M = \begin{pmatrix} S_{M-1} & O_{M-1} \\ O_{M-1} & E_{M-1} \end{pmatrix}$$

Then G^0 and G^1 will acquire the form

$$G^0 = \begin{pmatrix} E_{M-1} & O_{M-1} \\ O_{M-1} & -E_{M-1} \end{pmatrix}, \qquad G^1 = \begin{pmatrix} O_{M-1} & E_{M-1} \\ E_{M-1} & O_{M-1} \end{pmatrix}$$

From (6) it follows that the rest of the matrixes G^n will have the form

$$G^n = \begin{pmatrix} O_{M-1} & iG^n_{M-1} \\ -iG^n_{M-1} & O_{M-1} \end{pmatrix}, \qquad n \geq 2$$

where the matrixes G^n_{M-1} of order 2^{M-1} have to satisfy (6).

Therefore, the initial aggregate $\bar{H}_M$ is similar to the following aggregate of $2M$ matrixes:

$$\begin{pmatrix} E_{M-1} & O_{M-1} \\ O_{M-1} & -E_{M-1} \end{pmatrix}, \quad \begin{pmatrix} O_{M-1} & E_{M-1} \\ E_{M-1} & O_{M-1} \end{pmatrix}, \quad \begin{pmatrix} O_{M-1} & i\bar{H}_{M-1} \\ -i\bar{H}_{M-1} & O_{M-1} \end{pmatrix} \tag{8}$$

where, in order to satisfy (6), $2(M-1)$ matrixes $\bar{H}_{M-1}$ of order 2^{M-1} also have to fulfill correlation (6).

If we apply to aggregate (8) the similarity transformation having the transformation matrix

$$T_M = \begin{pmatrix} T_{M-1} & O_{M-1} \\ O_{M-1} & T_{M-1} \end{pmatrix}$$

then this aggregate will be transformed to the aggregate of the same form (8) in which only $\bar{H}_{M-1}$ is replaced by $T^{-1}_{M-1}\bar{H}_{M-1}T_{M-1}$. Therefore, the proof of the theorem for $2M$ matrixes $\bar{H}_M$ of order 2^M has been reduced to the same proof for $2(M-1)$ matrixes $\bar{H}_{M-1}$ of order 2^{M-1}.

Continuing this process and decreasing the order of the matrixes to 2, we finally obtain the result $\bar{H}_M \sim \bar{G}_M$ and $\bar{G}_M$ satisfies (6). Hence the theorem has been proved.

Let us note that to the aggregate of G^n, $0 \leq n \leq 2M-1$, we can add one more matrix $G^{2M} = G^0G^1 \ldots G^{2M-1}$ corresponding to (6).

It is easy to prove that the Hermitian conjugation $\bar{G}^+_M$ of the aggregate $\bar{G}_M$ of (7) has the following property:

$$\bar{G}^+_M = \bar{G}_M \tag{9}$$

For such aggregates of matrixes we shall prove a special theorem. This theorem, like the previous one, is a generalization of the well-known results for the Dirac matrixes (Bjorken and Drell, 1964).

Theorem 2. Every aggregate $\bar{H}_M$ of $2M$ matrixes of order 2^M fulfilling correlation (6) and (9), $\bar{H}^+_M = \bar{H}_M$, is related with $\bar{G}_M$ by the correlation

$$\bar{H}_M = T^+_M\bar{G}_MT_M, \qquad T^+_M = T^{-1}_M \tag{10}$$

Proof. First let us prove the following lemma.

Lemma. Any matrix S_M commutative with all of the matrixes of the aggregate $\bar{G}_M$ is proportional to the unit matrix: $S_M = \lambda E_M$.

In order to prove the lemma, let us present S_M in the form

$$S_M=\begin{pmatrix} S_{M-1}^{11} & S_{M-1}^{12} \\ S_{M-1}^{21} & S_{M-1}^{22} \end{pmatrix}$$

Then from its commutativity with the matrixes $\bar{G}_M$ of (7) we have

$$\begin{gathered} S_{M-1}^{12}=S_{M-1}^{21}=O_{M-1}, \\ S_{M-1}^{11}=S_{M-1}^{22}, \qquad S_{M-1}^{11}\bar{G}_{M-1}=\bar{G}_{M-1}S_{M-1}^{11} \end{gathered} \tag{11}$$

Repeating the same process for the matrix S_{M-1}^{11} and then for $S_{M-2}^{11}, S_{M-3}^{11}, \ldots, S_1^{11}$, we obtain the result that the matrix S_M is diagonal with the same numbers on its main diagonal, which was to be proved.

Let us return to the proof of the theorem. It follows from Theorem 1 that $\bar{H}_M=C_M^{-1}\bar{G}_M C_M$.

As $\bar{H}_M^+=\bar{H}_M$ and $\bar{G}_M^+=\bar{G}_M$ we have

$$\bar{H}_M^+=C_M^+\bar{G}_M(C_M^+)^{-1}=C_M^{-1}\bar{G}_M C_M=\bar{H}_M \tag{12}$$

From here we find

$$S_M\bar{G}_M=\bar{G}_M S_M, \qquad S_M=C_M C_M^+ \tag{13}$$

Therefore, from the proved lemma we obtain

$$C_M C_M^+=\lambda E_M \tag{14}$$

Since there are only nonnegative real numbers on the main diagonal of the matrix $C_M C_M^+$, the number λ is real and positive. Supposing $T_M=C_M/\sqrt{\lambda}$, we obtain (10). The theorem has been proved.

Let us turn now to the conservation law of the current which may be deduced from equation (1). In order to obtain it, we have to find a matrix R which satisfies

$$(R\Gamma^n)^+=R\Gamma^n \tag{15}$$

Then analogously to the Dirac theory we obtain the following for the current density J^n:

$$J^n=e\bar{\psi}\Gamma^n\psi, \qquad \partial_n J^n=0, \qquad \bar{\psi}=\psi^+R \tag{16}$$

In addition we must require that the current density J^n have a vector character:

$$J^n=\frac{\partial x^n}{\partial x^{k'}}J^{k'} \tag{17}$$

Let us consider the problem of finding the matrix R and the transformation law of the wave function ψ for the transition from coordinates x^k to the new coordinates $x^{k'}$.

On the matrixes G^k related to Γ^k by (6) we impose condition (9). This condition is fulfilled, for example, by the matrixes of the aggregate $\bar{G}_{2N}$ of (7) and it is necessary to make the values J^k real numbers.

We seek the transformation law for ψ in the form

$$\psi'(x^{k'}) = F\psi(x^k) \tag{18}$$

where F is a nonsingular matrix.

Then from the relativistic covariance of equation (1) we have

$$F^{-1}\Gamma^n \frac{\partial x^k}{\partial x^{n'}} F = \Gamma^k \tag{19}$$

As usual, let us consider the infinitesimal rotation

$$\frac{\partial x^k}{\partial x^{n'}} = \delta_n^k + \varepsilon h_{kn}, \qquad h_{kn} = -h_{nk}, \qquad \varepsilon \to 0, \qquad F = E + \varepsilon\Omega \tag{20}$$

Here δ_n^k is the Kronecker symbol, and E is the unit matrix.

Then we derive from (19)

$$[\Omega, \Gamma^k] = \Gamma^n h_{kn} \tag{21}$$

The correlation (21) has the following solution:

$$\Omega = \mu E + \Gamma^n \Gamma^l \lambda_{nl}, \qquad \lambda_{nl} = -\frac{g^{nn}}{4} h_{nl} \tag{22}$$

in which μ is an arbitrary number. It follows from the proved lemma and Theorem 1 that formula (22) gives us the general solution of (21).

From formulas (16)–(18) we have the following correlation:

$$F^+ R\Gamma^n \frac{\partial x^k}{\partial x^{n'}} F = R\Gamma^k \tag{23}$$

From (19) and (23) we find

$$F^+ R = RF^{-1} \tag{24}$$

Let us return to the infinitesimal rotation (20). Then from (20) and (24) we obtain

$$\Omega^+ R = -R\Omega \tag{25}$$

So we have to find the matrix R which satisfies (15) and (25). This gives the following correlations:

$$\begin{gathered} \mu = i\nu, \qquad \operatorname{Im} \nu = 0, \qquad (R\Gamma^n)^+ = R\Gamma^n, \\ (\Gamma^k \Gamma^l)^+ R = -R\Gamma^k \Gamma^l, \qquad k \neq l \end{gathered} \tag{26}$$

Using (6) and (9), it is not difficult to check the following solution of (26):

$$R = si^{N(N+1)/2+1} \prod_{k=0}^{N-1} \Gamma^{4k}, \qquad \mathrm{Im}\, s = 0 \tag{27}$$

For rotation by an angle θ_{kl} in the plane $x^k x^l$ the transformation formula for ψ acquires the well-known form

$$\psi' = \exp[i\alpha - \tfrac{1}{2} g^{kk} \Gamma^k \Gamma^l \theta_{kl}]\psi, \qquad \mathrm{Im}\, \alpha = 0 \tag{28}$$

in which k, l are fixed numbers.

It ensues from (19) and (24) that the expressions

$$A^{l_1 l_2 \ldots l_n} = \bar{\psi} \Gamma^{l_1} \Gamma^{l_2} \ldots \Gamma^{l_n} \psi$$

are tensors.

Theorem 2 gives us the result that these tensors and in particular the current densities J^n are the same, independent of the choice of the matrixes Γ^n satisfying (9).

From the differential law of the current conservation (16) we have the following N continuity equations:

$$\frac{\partial (I^{k0} + V^{k0})}{c\, \partial t} + \frac{\partial I^{k\alpha}}{\partial x^{\alpha}_{(k)}} = 0, \qquad \alpha = 1, 2, 3 \tag{29}$$

where

$$I^{kn} = \int J^{4(k-1)+n} \prod_{l=1, l \neq k}^{N} d^3 x_{(l)}, \qquad V^{k0} = \sum_{j=1, j \neq k}^{N} I^{j0} \tag{30}$$

The $x^n_{(l)}$ are the four coordinates of the lth particle, $x^0_{(l)} = ct$, $1 \leq l \leq N$.

The N equations (29) give us only one integral conservation law.

The quantities I^{k0} and $I^{k\alpha}$ can be interpreted as the charge and current densities of the kth particle, and V^{k0} as the charge flow in the kth particle from the other particles.

As a three-dimensional volume is not a relativistic invariant, the expressions I^{kn} are not vectors. That is why we introduce other quantities $\bar{I}^{kn}$ which are vectors and which coincide with I^{kn} in the nonrelativistic case:

$$\bar{I}^{kn}(\tau, x^{\alpha}_{(k)}) = \int J^{4(k-1)+n}(\tau, x^{\alpha}_{(l)}) \times \prod_{l=1, l \neq k}^{N} \frac{d^3 x_{(l)}}{(1 - v_l^2/c^2)^{1/2}}, \qquad \alpha = 1, 2, 3 \tag{31}$$

where τ is the proper time and is the same under integration for all the particles; $v_l = v_l(\tau, x^{\alpha}_{(l)})$ are the components of the lth particle velocity. This velocity can be determined by imposing the following condition: In the inertial coordinate system moving with this velocity relative to the inertial system x^n the correlations $I^{l\beta}(\tau, x^{\alpha}_{(l)}) = 0$, $\beta = 1, 2, 3$, must be fulfilled.

Now we can set up the equation for the potentials $A^{kn}_{\text{int}}(x^n_{(k)})$ $(1 \le k \le N, 0 \le n \le 3)$ of the interaction of the kth particle with the other particles, which will coincide with the classical equation in the nonrelativistic case:

$$\frac{\partial}{\partial x^l_{(k)}} \frac{\partial}{\partial x_{(k)l}} A^{kn}_{\text{int}} - \frac{\partial^2 A^{kl}_{\text{int}}}{\partial x^l_{(k)}\, \partial x_{(k)n}} = \frac{4\pi}{c} \bar{V}^{kn}$$
$$\bar{V}^{kn} = \sum_{l=1, l\neq k}^{N} \bar{I}^{ln} \tag{32}$$

Let us consider what happens when there are two electrons. Then for the matrix representation (7), equation (1) has the form

$$\begin{aligned}
&i\hbar[\bar{\partial}_{x^0}\tilde{\psi}_1 - (\bar{\partial}_{x^2} - i\bar{\partial}_{x^1})\tilde{\psi}_3 - \bar{\partial}_{x^3}\tilde{\psi}_4 + \gamma^n\bar{\partial}_{y^n}\tilde{\psi}_4] - \sqrt{2}\, mc\tilde{\psi}_1 = 0\\
&i\hbar[\bar{\partial}_{x^0}\tilde{\psi}_2 + (\bar{\partial}_{x^2} + i\bar{\partial}_{x^1})\tilde{\psi}_4 - \bar{\partial}_{x^3}\tilde{\psi}_3 - \gamma^n\bar{\partial}_{y^n}\tilde{\psi}_3] - \sqrt{2}\, mc\tilde{\psi}_2 = 0\\
&i\hbar[-\bar{\partial}_{x^0}\tilde{\psi}_3 + (\bar{\partial}_{x^2} + i\bar{\partial}_{x^1})\tilde{\psi}_1 + \bar{\partial}_{x^3}\tilde{\psi}_2 - \gamma^n\bar{\partial}_{y^n}\tilde{\psi}_2] - \sqrt{2}\, mc\tilde{\psi}_3 = 0\\
&i\hbar[-\bar{\partial}_{x^0}\tilde{\psi}_4 - (\bar{\partial}_{x^2} - i\bar{\partial}_{x^1})\tilde{\psi}_2 + \bar{\partial}_{x^3}\tilde{\psi}_1 + \gamma^n\bar{\partial}_{y^n}\tilde{\psi}_1] - \sqrt{2}\, mc\tilde{\psi}_4 = 0
\end{aligned} \tag{33}$$

Here $\tilde{\psi}_k$ denotes the four-component functions with the components ψ_{4k-3}, ψ_{4k-2}, ψ_{4k-1}, ψ_{4k}; $\bar{\partial}_{x^k} = \partial/\partial x^k + (ie/c\hbar)A_k$ and $\bar{\partial}_{y^k} = \partial/\partial y^k + (ie/c\hbar)A_{4+k}$; x^k, y^k are the coordinates of the two electrons; γ^n are the Dirac matrixes.

Let us consider the nonrelativistic case and present $\tilde{\psi}_k$ in the form

$$\tilde{\psi}_k = \exp\left[-\frac{i}{\hbar} mc(x^0 + y^0)\right]\tilde{\varphi}_k \tag{34}$$

Then from (33) using the nonrelativistic approach, we obtain the following approximate correlations:

$$(1 - \sqrt{2})\tilde{\varphi}_1 = -\gamma^0\tilde{\varphi}_4, \qquad (1 - \sqrt{2})\tilde{\varphi}_2 = \gamma^0\tilde{\varphi}_3 \tag{35}$$

$$(1 + \sqrt{2})\tilde{\varphi}_3 = -\gamma^0\tilde{\varphi}_2, \qquad (1 + \sqrt{2})\tilde{\varphi}_4 = \gamma^0\tilde{\varphi}_1 \tag{36}$$

Let us note that the correlations (36) follow from (35).

From (7), (16), (35), and (36) we find the following expression for the charge density of the two electrons:

$$\begin{aligned} J^0 &= \tilde{\varphi}_1^+(\tilde{\varphi}_1 + \gamma^0\tilde{\varphi}_4) + \tilde{\varphi}_2^+(\tilde{\varphi}_2 - \gamma^0\tilde{\varphi}_3) \\ &\quad - \tilde{\varphi}_3^+(\gamma^0\tilde{\varphi}_2 + \tilde{\varphi}_3) + \tilde{\varphi}_4^+(\gamma^0\tilde{\varphi}_1 - \tilde{\varphi}_4) \\ &= \sqrt{2}\,(\tilde{\varphi}_1^+\tilde{\varphi}_1 + \tilde{\varphi}_2^+\tilde{\varphi}_2 + \tilde{\varphi}_3^+\tilde{\varphi}_3 + \tilde{\varphi}_4^+\tilde{\varphi}_4) \end{aligned} \tag{37}$$

Hence in the nonrelativistic case under consideration the charge density J^0 is a positive quantity.

Let us consider a nonrelativistic stationary case having only an external magnetic field intensity: $F_{12} = F_{56} = H_z$ (the other magnetic field intensities are considered small). Then from (3) and (4) we obtain approximately the following:

$$\begin{aligned} &\left[\mathscr{E}_1 + \mathscr{E}_2 - e(A_0 + A_4) - \frac{\hbar^2}{2m}\left[\left(\partial^\alpha + \frac{ie}{c\hbar}A^\alpha\right)\left(\partial_\alpha + \frac{ie}{c\hbar}A_\alpha\right)\right.\right. \\ &\quad \left. + \left(\partial^{4+\alpha} + \frac{ie}{c\hbar}A^{4+\alpha}\right)\left(\partial_{4+\alpha} + \frac{ie}{c\hbar}A_{4+\alpha}\right)\right] \\ &\quad \left. - \frac{i\hbar e}{2mc}(\Gamma^1\Gamma^2 + \Gamma^5\Gamma^6)H_z\right]\varphi = 0, \qquad \alpha = 1, 2, 3 \end{aligned} \tag{38}$$

The matrix $\frac{1}{2}i\hbar(\Gamma^1\Gamma^2 + \Gamma^5\Gamma^6)$ gives for the summary spin projection S_z for the two electrons the values $S_z = -\hbar, \hbar, 0$, which coincide with the classical result.

REFERENCES

Bjorken, J. D., and Drell, S. D. (1964). *Relativistic Quantum Mechanics*, McGraw-Hill, New York, Vol. 1.

Rashevsky, P. K. (1955). *Uspekhi Matematicheskikh Nauk*, X(2), 3–110.